HL Paper 1

Consider the function $f : x \to \sqrt{\frac{\pi}{4} - \arccos x}$.

(a) Find the largest possible domain of *f*.

(b) Determine an expression for the inverse function, f^{-1}, and write down its domain.

A function f is defined by $f(x) = \frac{3x-2}{2x-1},\ x \in \mathbb{R},\ x \ne \frac{1}{2}$.

a. Find an expression for $f^{-1}(x)$. [4]

b. Given that $f(x)$ can be written in the form $f(x) = A + \frac{B}{2x-1}$, find the values of the constants A and B. [2]

c. Hence, write down $\int \frac{3x-2}{2x-1}\mathrm{d}x$. [1]

Write $\ln(x^2 - 1) - 2\ln(x + 1) + \ln(x^2 + x)$ as a single logarithm, in its simplest form.

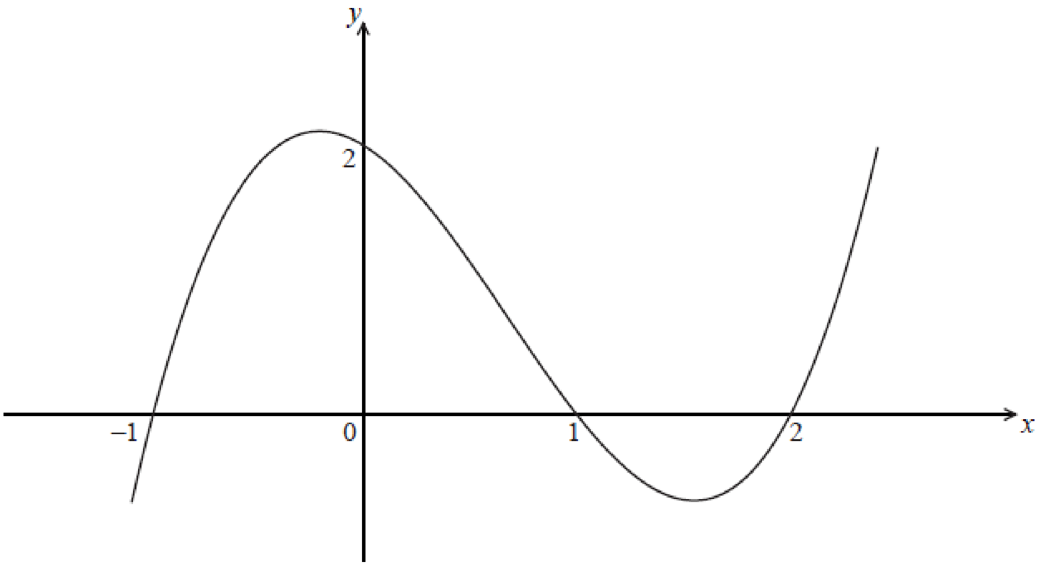
Consider the equation $yx^2 + (y - 1)x + (y - 1) = 0$.

a. Find the set of values of *y* for which this equation has real roots. [4]

b. Hence determine the range of the function $f : x \to \frac{x+1}{x^2+x+1}$. [3]

c. Explain why *f* has no inverse. [1]

Let $f(x) = x^3 + ax^2 + bx + c$, where a, b, $c \in \mathbb{Z}$. The diagram shows the graph of $y = f(x)$.



a. Using the information shown in the diagram, find the values of a , b and c . [4]

b. If $g(x) = 3f(x - 2)$, [3]

(i) state the coordinates of the points where the graph of g intercepts the x-axis.
(ii) Find the y-intercept of the graph of g .

The function f is defined by $f(x) = \frac{3x}{x-2},\ x \in \mathbb{R},\ x \neq 2.$

a. Sketch the graph of $y = f(x)$, indicating clearly any asymptotes and points of intersection with the x and y axes. [4]

b. Find an expression for $f^{-1}(x)$. [4]

c. Find all values of x for which $f(x) = f^{-1}(x)$. [3]

d. Solve the inequality $|f(x)| < \frac{3}{2}$. [4]

e. Solve the inequality $f(|x|) < \frac{3}{2}$. [2]

A function is defined by $h(x) = 2e^x - \frac{1}{e^x},\ x \in \mathbb{R}$. Find an expression for $h^{-1}(x)$.

The polynomial $P(x) = x^3 + ax^2 + bx + 2$ is divisible by $(x + 1)$ and by $(x - 2)$.

Find the value of a and of b, where $a,\ b \in \mathbb{R}$.

Consider the function f defined by $f(x) = x^2 - a^2,\ x \in \mathbb{R}$ where a is a positive constant.

The function g is defined by $g(x) = x\sqrt{f(x)}$ for $|x| > a$.

a.i. Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [2]

$y = f(x)$;

a.ii. Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [4]

$y = \frac{1}{f(x)}$;

a.iii. Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [2]

$y = \left|\frac{1}{f(x)}\right|$.

b. Find $\int f(x) \cos x \mathrm{d}x$. [5]

c. By finding $g'(x)$ explain why g is an increasing function. [4]

The functions f and g are defined by $f(x) = 2x + \frac{\pi}{5},\ x \in \mathbb{R}$ and $g(x) = 3\sin x + 4,\ x \in \mathbb{R}$.

a. Show that $g \circ f(x) = 3\sin\left(2x + \frac{\pi}{5}\right) + 4$. [1]

b. Find the range of $g \circ f$. [2]

c. Given that $g \circ f\left(\frac{3\pi}{20}\right) = 7$, find the next value of x, greater than $\frac{3\pi}{20}$, for which $g \circ f(x) = 7$. [2]

d. The graph of $y = g \circ f(x)$ can be obtained by applying four transformations to the graph of $y = \sin x$. State what the four transformations represent geometrically and give the order in which they are applied. [4]

Consider the functions $f(x) = \tan x,\ 0 \le\ x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \ne 1$.

a. Find an expression for $g \circ f(x)$, stating its domain. [2]

b. Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. [2]

c. Let $y = g \circ f(x)$, find an exact value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. [6]

d. Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. [6]

Consider the function $f(x) = \frac{1}{x^2+3x+2},\ x \in \mathbb{R},\ x \neq -2,\ x \neq -1.$

a.i. Express $x^2 + 3x + 2$ in the form $(x+h)^2 + k$. [1]

a.ii. Factorize $x^2 + 3x + 2$. [1]

b. Sketch the graph of $f(x)$, indicating on it the equations of the asymptotes, the coordinates of the y-intercept and the local maximum. [5]

c. Show that $\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{x^2+3x+2}$. [1]

d. Hence find the value of p if $\int_0^1 f(x)\mathrm{d}x = \ln(p)$. [4]

e. Sketch the graph of $y = f\left(|x|\right)$. [2]

f. Determine the area of the region enclosed between the graph of $y = f\left(|x|\right)$, the x-axis and the lines with equations $x = -1$ and $x = 1$. [3]

Let $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36,\ x \in \mathbb{R}$.

a. For the polynomial equation $p(x) = 0$, state [3]

(i) the sum of the roots;

(ii) the product of the roots.

b. A new polynomial is defined by $q(x) = p(x+4)$. [2]

Find the sum of the roots of the equation $q(x) = 0$.

Let $y(x) = xe^{3x},\ x \in \mathbb{R}$.

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. [7]

c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. [5]

Justify whether any such point is a maximum or a minimum.

d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. [5]

e. Hence sketch the graph of $y(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [2]

The quadratic equation $x^2 - 2kx + (k-1) = 0$ has roots α and β such that $\alpha^2 + \beta^2 = 4$. Without solving the equation, find the possible values of the real number k.

A given polynomial function is defined as $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. The roots of the polynomial equation $f(x) = 0$ are consecutive terms of a geometric sequence with a common ratio of $\frac{1}{2}$ and first term 2.

Given that $a_{n-1} = -63$ and $a_n = 16$ find

a. the degree of the polynomial; [4]

b. the value of a_0. [2]

Consider the function f, where $f(x) = \arcsin(\ln x)$.

(a) Find the domain of f.

(b) Find $f^{-1}(x)$.

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a. Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0,\ n \in \mathbb{Z}$. [2]

b. Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$. [2]

c. Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2,\ n \in \mathbb{Z}$. [9]

Let $g(x) = \log_5 |2\log_3 x|$. Find the product of the zeros of g .

The functions f and g are defined by $f(x) = ax^2 + bx + c,\ x \in \mathbb{R}$ and $g(x) = p\sin x + qx + r,\ x \in \mathbb{R}$ where $a,\ b,\ c,\ p,\ q,\ r$ are real constants.

a. Given that f is an even function, show that $b = 0$. [2]

b. Given that g is an odd function, find the value of r. [2]

c. The function h is both odd and even, with domain $\mathbb{R}$. [2]

Find $h(x)$.

Consider the polynomial $q(x) = 3x^3 - 11x^2 + kx + 8$.

a. Given that $q(x)$ has a factor $(x - 4)$, find the value of k. [3]

b. Hence or otherwise, factorize $q(x)$ as a product of linear factors. [3]

Consider the polynomial $P\left(z\right) = z^5 - 10z^2 + 15z - 6,\ z \in \mathbb{C}$.

The polynomial can be written in the form $P(z) = (z - 1)^3(z^2 + bz + c)$.

Consider the function $q\left(x\right) = x^5 - 10x^2 + 15x - 6,\ x \in \mathbb{R}$.

a. Write down the sum and the product of the roots of $P(z) = 0$. [2]

b. Show that $(z - 1)$ is a factor of $P(z)$. [2]

c. Find the value of b and the value of c. [5]

d. Hence find the complex roots of $P(z) = 0$. [3]

e.i. Show that the graph of $y = q(x)$ is concave up for $x > 1$. [3]

e.ii. Sketch the graph of $y = q(x)$ showing clearly any intercepts with the axes. [3]

The function f is defined as $f(x) = \frac{3x+2}{x+1},\ x \in \mathbb{R},\ x \neq -1$.

Sketch the graph of $y = f(x)$, clearly indicating and stating the equations of any asymptotes and the coordinates of any axes intercepts.

The cubic equation $x^3 + px^2 + qx + c = 0$, has roots $\alpha,\ \beta,\ \gamma$. By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that

a. (i) $p = -(\alpha + \beta + \gamma)$; [3]

(ii) $q = \alpha\beta + \beta\gamma + \gamma\alpha$;

(iii) $c = -\alpha\beta\gamma$.

b. It is now given that $p = -6$ and $q = 18$ for parts (b) and (c) below. [5]

(i) In the case that the three roots $\alpha,\ \beta,\ \gamma$ form an arithmetic sequence, show that one of the roots is 2.

(ii) Hence determine the value of c.

c. In another case the three roots $\alpha,\ \beta,\ \gamma$ form a geometric sequence. Determine the value of c. [6]

Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$
$g(x) = \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$

a. Sketch the graph of $y = h(x)$. [2]

b. Find an expression for the composite function $h \circ g(x)$ and state its domain. [2]

c. Given that $f(x) = h(x) + h \circ g(x)$, [7]

(i) find $f'(x)$ in simplified form;
(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

d. Nigel states that f is an odd function and Tom argues that f is an even function. [3]

(i) State who is correct and justify your answer.
(ii) Hence find the value of $f(x)$ for $x < 0$.

The function f is defined by $f(x) = 2x^3 + 5,\ \ -2 \leqslant x \leqslant 2$.

a. Write down the range of f. [2]

b. Find an expression for $f^{-1}(x)$. [2]

c. Write down the domain and range of f^{-1}. [2]

Consider the function defined by $f(x) = x\sqrt{1 - x^2}$ on the domain $-1 \leq x \leq 1$.

a. Show that f is an odd function. [2]

b. Find $f'(x)$. [3]

c. Hence find the x-coordinates of any local maximum or minimum points. [3]

d. Find the range of f. [3]

e. Sketch the graph of $y = f(x)$ indicating clearly the coordinates of the x-intercepts and any local maximum or minimum points. [3]

f. Find the area of the region enclosed by the graph of $y = f(x)$ and the x-axis for $x \geq 0$. [4]

g. Show that $\int_{-1}^{1} \left|x\sqrt{1-x^2}\right| \mathrm{d}x > \left|\int_{-1}^{1} x\sqrt{1-x^2}\mathrm{d}x\right|$. [2]

The function f is defined by

$$f(x) = \begin{cases} 2x - 1, & x \leqslant 2 \\ ax^2 + bx - 5, & 2 < x < 3 \end{cases}$$

where a , $b \in \mathbb{R}$.

a. Given that f and its derivative, f' , are continuous for all values in the domain of f, find the values of a and b . [6]

b. Show that f is a one-to-one function. [3]

c. Obtain expressions for the inverse function f^{-1} and state their domains. [5]

The function f is defined by

$$f(x) = \begin{cases} 1 - 2x, & x \leq 2 \\ \frac{3}{4}(x-2)^2 - 3, & x > 2 \end{cases}$$

a. Determine whether or not f is continuous. [2]

b. The graph of the function g is obtained by applying the following transformations to the graph of f: [4]

a reflection in the y–axis followed by a translation by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Find $g(x)$.

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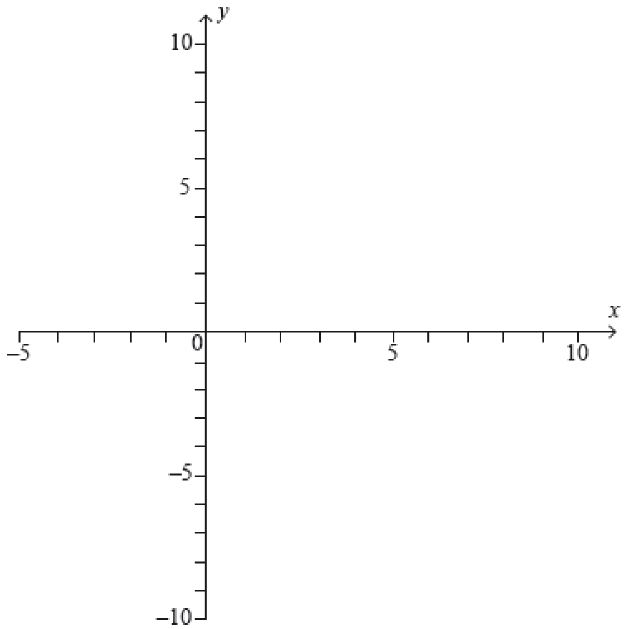
a. Sketch the graph of $y = \frac{1-3x}{x-2}$, showing clearly any asymptotes and stating the coordinates of any points of intersection with the axes. [4]



b. Hence or otherwise, solve the inequality $\left|\frac{1-3x}{x-2}\right| < 2$. [5]

The function f is given by $f(x) = x\mathrm{e}^{-x}$ $(x \geqslant 0)$.

a(i)(ii) Find an expression for $f'(x)$. [3]

(ii) Hence determine the coordinates of the point A, where $f'(x) = 0$.

b. Find an expression for $f''(x)$ and hence show the point A is a maximum. [3]

c. Find the coordinates of B, the point of inflexion. [2]

d. The graph of the function g is obtained from the graph of f by stretching it in the x-direction by a scale factor 2. [5]

(i) Write down an expression for $g(x)$.

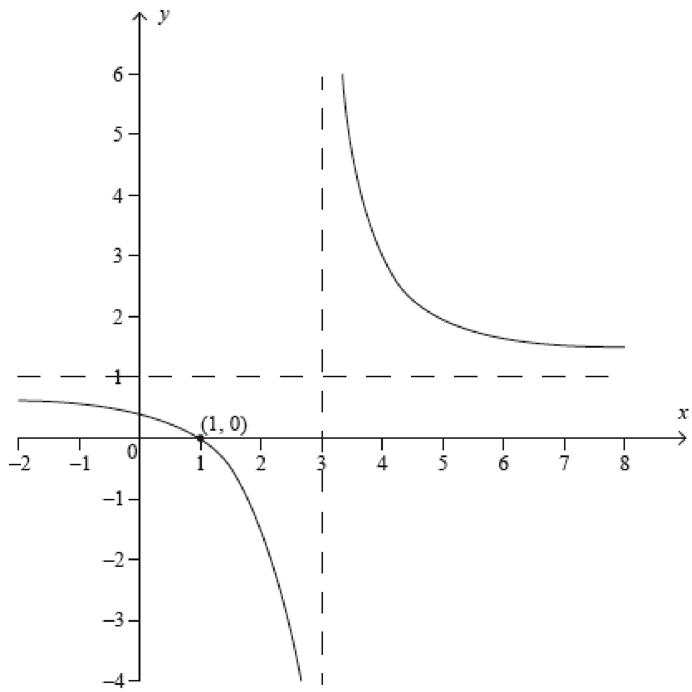
(ii) State the coordinates of the maximum C of g.

(iii) Determine the x-coordinates of D and E, the two points where $f(x) = g(x)$.

e. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, showing clearly the points A, B, C, D and E. [4]

f. Find an exact value for the area of the region bounded by the curve $y = g(x)$, the x-axis and the line $x = 1$. [3]

A rational function is defined by $f(x) = a + \frac{b}{x-c}$ where the parameters $a,\ b,\ c \in \mathbb{Z}$ and $x \in \mathbb{R}\backslash\{c\}$. The following diagram represents the graph of $y = f(x)$.



Using the information on the graph,

a. state the value of a and the value of c; [2]

b. find the value of b. [2]

The equation $5x^3 + 48x^2 + 100x + 2 = a$ has roots r_1, r_2 and r_3.

Given that $r_1 + r_2 + r_3 + r_1 r_2 r_3 = 0$, find the value of a.

Consider the equation $9x^3 - 45x^2 + 74x - 40 = 0$.

a. Write down the numerical value of the sum and of the product of the roots of this equation. [1]

b. The roots of this equation are three consecutive terms of an arithmetic sequence. [6]

Taking the roots to be α , $\alpha \pm \beta$, solve the equation.

Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants.

The remainder when $f(x)$ is divided by $(x + 1)$ is 7, and the remainder when $f(x)$ is divided by $(x - 2)$ is 1. Find the value of p and the value of q.

The cubic polynomial $3x^3 + px^2 + qx - 2$ has a factor $(x + 2)$ and leaves a remainder 4 when divided by $(x + 1)$. Find the value of p and the value of q.

The quadratic equation $2x^2 - 8x + 1 = 0$ has roots α and β.

a. Without solving the equation, find the value of [2]

(i) $\alpha + \beta$;

(ii) $\alpha\beta$.

b. Another quadratic equation $x^2 + px + q = 0,\ p,\ q \in \mathbb{Z}$ has roots $\frac{2}{\alpha}$ and $\frac{2}{\beta}$. [4]

Find the value of p and the value of q.

Consider the following functions:

$$f(x) = \frac{2x^2 + 3}{75},\ x \geqslant 0$$

$$g(x) = \frac{|3x - 4|}{10},\ x \in \mathbb{R}\ .$$

a. State the range of f and of g . [2]

b. Find an expression for the composite function $f \circ g(x)$ in the form $\frac{ax^2+bx+c}{3750}$, where $a,\ b$ and $c \in \mathbb{Z}$. [4]

c. (i) Find an expression for the inverse function $f^{-1}(x)$. [4]

(ii) State the domain and range of f^{-1}.

d. The domains of f and g are now restricted to $\{0, 1, 2, 3, 4\}$. [6]

By considering the values of f and g on this new domain, determine which of f and g could be used to find a probability distribution for a discrete random variable X, stating your reasons clearly.

e. Using this probability distribution, calculate the mean of X. [2]

Consider the graphs of $y = |x|$ and $y = -|x| + b$, where $b \in \mathbb{Z}^+$.

a. Sketch the graphs on the same set of axes. [2]

b. Given that the graphs enclose a region of area 18 square units, find the value of *b*. [3]

Consider the functions given below.

$$f(x) = 2x + 3$$

$$g(x) = \frac{1}{x}, x \neq 0$$

a. (i) Find $(g \circ f)(x)$ and write down the domain of the function. [2]

(ii) Find $(f \circ g)(x)$ and write down the domain of the function.

b. Find the coordinates of the point where the graph of $y = f(x)$ and the graph of $y = \left(g^{-1} \circ f \circ g\right)(x)$ intersect. [4]

Consider the function $f_n(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^n x),\ n \in \mathbb{Z}^+$.

a. Determine whether f_n is an odd or even function, justifying your answer. [2]

b. By using mathematical induction, prove that [8]

$$f_n(x) = \frac{\sin 2^{n+1}x}{2^n \sin 2x},\ x \neq \frac{m\pi}{2} \text{ where } m \in \mathbb{Z}.$$

c. Hence or otherwise, find an expression for the derivative of $f_n(x)$ with respect to x. [3]

d. Show that, for $n > 1$, the equation of the tangent to the curve $y = f_n(x)$ at $x = \frac{\pi}{4}$ is $4x - 2y - \pi = 0$. [8]

A function f is defined by $f(x) = \frac{2x-3}{x-1},\ x \neq 1$.

(a) Find an expression for $f^{-1}(x)$.

(b) Solve the equation $\left|f^{-1}(x)\right| = 1 + f^{-1}(x)$.

The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

a. Find the value of p and the value of q . [4]

b. The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph. [2]

Consider a function f, defined by $f(x) = \frac{x}{2-x}$ for $0 \leqslant x \leqslant 1$.

a. Find an expression for $(f \circ f)(x)$. [3]

b. Let $F_n(x) = \frac{x}{2^n - (2^n - 1)x}$, where $0 \leqslant x \leqslant 1$. [8]
Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$$

.

c. Show that $F_{-n}(x)$ is an expression for the inverse of F_n . [6]

d. (i) State $F_n(0)$ and $F_n(1)$. [6]
(ii) Show that $F_n(x) < x$, given $0 < x < 1$, $n \in \mathbb{Z}^+$.
(iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1} , the x-axis and the line $x = 1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n .

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a. State the set of values of a for which the function $x \mapsto \log_a x$ exists, for all $x \in \mathbb{R}^+$. [2]

b. Given that $\log_x y = 4\log_y x$, find all the possible expressions of y as a function of x. [6]

When the function $q(x) = x^3 + kx^2 - 7x + 3$ is divided by $(x + 1)$ the remainder is seven times the remainder that is found when the function is divided by $(x + 2)$.

Find the value of k .

Solve $(\ln x)^2 - (\ln 2)(\ln x) < 2(\ln 2)^2$.

The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β.

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$;

(b) find a quadratic equation with roots α^2 and β^2.

When the polynomial $3x^3 + ax + b$ is divided by $(x - 2)$, the remainder is 2, and when divided by $(x + 1)$, it is 5. Find the value of a and the value of b.

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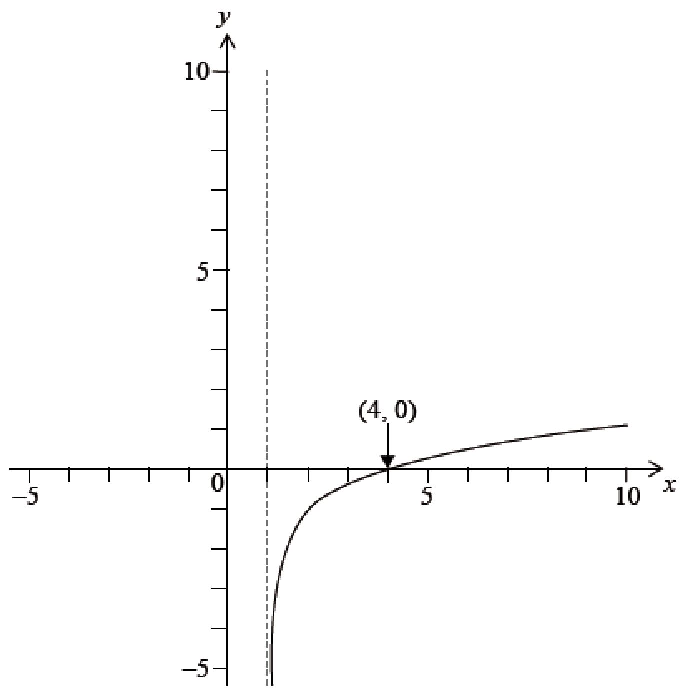
a. Sketch on the same axes the curve $y = \left|\frac{7}{x-4}\right|$ and the line $y = x + 2$, clearly indicating any axes intercepts and any asymptotes. [3]

b. Find the exact solutions to the equation $x + 2 = \left|\frac{7}{x-4}\right|$. [5]

The diagram below shows a sketch of the graph of $y = f(x)$.



a. Sketch the graph of $y = f^{-1}(x)$ on the same axes. [2]

b. State the range of f^{-1}. [1]

c. Given that $f(x) = \ln(ax + b),\ x > 1$, find the value of a and the value of b. [4]

Find the set of values of x for which $|x - 1| > |2x - 1|$.

Let $f(x) = \frac{4}{x+2},\ x \neq -2$ and $g(x) = x - 1$.

If $h = g \circ f$, find

(a) $h(x)$;

(b) $h^{-1}(x)$, where h^{-1} is the inverse of h.

The function f is given by $f(x) = \frac{3^x+1}{3^x-3^{-x}}$, for $x > 0$.

a. Show that $f(x) > 1$ for all $x > 0$. [3]

b. Solve the equation $f(x) = 4$. [4]

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a. (i) Express each of the complex numbers $z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and $z_3 = -2\mathrm{i}$ in modulus-argument form. [9]

(ii) Hence show that the points in the complex plane representing z_1, z_2 and z_3 form the vertices of an equilateral triangle.

(iii) Show that $z_1^{3n} + z_2^{3n} = 2z_3^{3n}$ where $n \in \mathbb{N}$.

b. (i) State the solutions of the equation $z^7 = 1$ for $z \in \mathbb{C}$, giving them in modulus-argument form. [9]

(ii) If w is the solution to $z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of π.

(iii) Show that $z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial $z^7 - 1$. State the two other quadratic factors with real coefficients.

Given the complex numbers $z_1 = 1 + 3\mathrm{i}$ and $z_2 = -1 - \mathrm{i}$.

a. Write down the exact values of $|z_1|$ and $\arg(z_2)$. [2]

b. Find the minimum value of $|z_1 + \alpha z_2|$, where $\alpha \in \mathbb{R}$. [5]

The same remainder is found when $2x^3 + kx^2 + 6x + 32$ and $x^4 - 6x^2 - k^2x + 9$ are divided by $x + 1$. Find the possible values of k .

When $3x^5 - ax + b$ is divided by $x - 1$ and $x + 1$ the remainders are equal. Given that a , $b \in \mathbb{R}$, find

(a) the value of a ;

(b) the set of values of b .

(a) Express the quadratic $3x^2 - 6x + 5$ in the form $a(x+b)^2 + c$, where $a, b, c \in \mathbb{Z}$.

(b) Describe a sequence of transformations that transforms the graph of $y = x^2$ to the graph of $y = 3x^2 - 6x + 5$.

The function f is defined by $f(x) = \frac{1}{x}$, $x \neq 0$.

The graph of the function $y = g(x)$ is obtained by applying the following transformations to the graph of $y = f(x)$:

a translation by the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$; a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

a. Find an expression for $g(x)$. [2]

b. State the equations of the asymptotes of the graph of g. [2]

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a. Factorize $z^3 + 1$ into a linear and quadratic factor. [2]

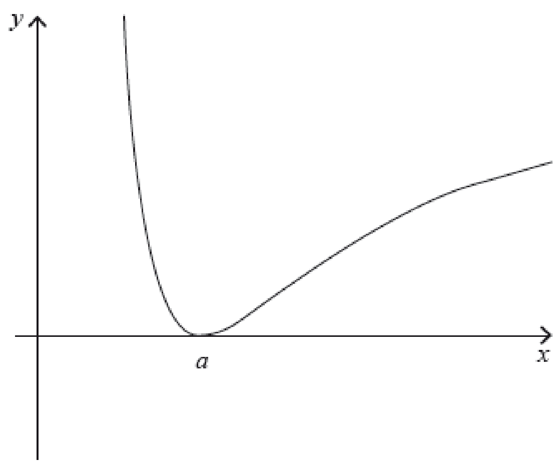
b. Let $\gamma = \frac{1+\mathrm{i}\sqrt{3}}{2}$. [9]

(i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$.

The following diagram shows the graph of $y = \frac{(\ln x)^2}{x}$, $x > 0$.



The region R is enclosed by the curve, the x-axis and the line $x = e$.

Let $I_n = \int_1^{\mathrm{e}} \frac{(\ln x)^n}{x^2} \mathrm{d}x,\ n \in \mathbb{N}$.

a. Given that the curve passes through the point $(a,\ 0)$, state the value of a. [1]

b. Use the substitution $u = \ln x$ to find the area of the region R. [5]

c. (i) Find the value of I_0. [7]

(ii) Prove that $I_n = \frac{1}{\mathrm{e}} + nI_{n-1},\ n \in \mathbb{Z}^+$.

(iii) Hence find the value of I_1.

d. Find the volume of the solid formed when the region R is rotated through 2π about the x-axis. [5]

Given that $Ax^3 + Bx^2 + x + 6$ is exactly divisible by $(x+1)(x-2)$, find the value of A and the value of B .

The function f is defined, for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, by $f(x) = 2\cos x + x\sin x$.

a. Determine whether f is even, odd or neither even nor odd. [3]

b. Show that $f''(0) = 0$. [2]

c. John states that, because $f''(0) = 0$, the graph of f has a point of inflexion at the point (0, 2) . Explain briefly whether John's statement is correct or not. [2]

---

a. Sketch the graph of $y = \left|\cos\left(\frac{x}{4}\right)\right|$ for $0 \leqslant x \leqslant 8\pi$. [2]

b. Solve $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$ for $0 \leqslant x \leqslant 8\pi$. [3]

The functions f and g are defined as:

$$f(x) = \mathrm{e}^{x^2},\ x \geqslant 0$$

$$g(x) = \frac{1}{x+3},\ x \neq -3.$$

(a) Find $h(x)$ where $h(x) = g \circ f(x)$.

(b) State the domain of $h^{-1}(x)$.

(c) Find $h^{-1}(x)$.

When $f(x) = x^4 + 3x^3 + px^2 - 2x + q$ is divided by $(x - 2)$ the remainder is 15, and $(x + 3)$ is a factor of $f(x)$.

Find the values of p and q .

Solve the following equations:

(a) $\log_2(x-2) = \log_4(x^2 - 6x + 12)$;

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$.

The function f is defined by $f(x) = \frac{ax+b}{cx+d}$, for $x \in \mathbb{R},\ x \neq -\frac{d}{c}$.

The function g is defined by $g(x) = \frac{2x-3}{x-2},\ x \in \mathbb{R},\ x \neq 2$

a. Find the inverse function f^{-1}, stating its domain. [5]

b.i. Express $g(x)$ in the form $A + \frac{B}{x-2}$ where A, B are constants. [2]

b.ii. Sketch the graph of $y = g(x)$. State the equations of any asymptotes and the coordinates of any intercepts with the axes. [3]

c. The function h is defined by $h(x) = \sqrt{x}$, for $x \geq 0$. [4]

State the domain and range of $h \circ g$.

The function f is defined by $f(x) = \frac{1}{4x^2-4x+5}$.

a. Express $4x^2 - 4x + 5$ in the form $a(x-h)^2 + k$ where $a, h, k \in \mathbb{Q}$. [2]

b. The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear. [3]

c. Sketch the graph of $y = f(x)$. [2]

d. Find the range of f. [2]

e. By using a suitable substitution show that $\int f(x)\mathrm{d}x = \frac{1}{4}\int \frac{1}{u^2+1}\mathrm{d}u$. [3]

f. Prove that $\int_1^{3.5} \frac{1}{4x^2-4x+5}\mathrm{d}x = \frac{\pi}{16}$. [7]

The function f is defined by $f(x) = \frac{2x-1}{x+2}$, with domain $D = \{x : -1 \leqslant x \leqslant 8\}$.

a. Express $f(x)$ in the form $A + \frac{B}{x+2}$, where A and $B \in \mathbb{Z}$. [2]

b. Hence show that $f'(x) > 0$ on D. [2]

c. State the range of f. [2]

d. (i) Find an expression for $f^{-1}(x)$. [8]

(ii) Sketch the graph of $y = f(x)$, showing the points of intersection with both axes.

(iii) On the same diagram, sketch the graph of $y = f'(x)$.

e. (i) On a different diagram, sketch the graph of $y = f(|x|)$ where $x \in D$. [7]

(ii) Find all solutions of the equation $f(|x|) = -\frac{1}{4}$.

The function f is defined as $f(x) = \mathrm{e}^{3x+1},\ x \in \mathbb{R}$.

a. (i) Find $f^{-1}(x)$. [4]

(ii) State the domain of f^{-1}.

b. The function g is defined as $g(x) = \ln x,\ x \in \mathbb{R}^{+}$. [5]

The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

Find the coordinates of P.

c. The graph of $y = g(x)$ intersects the x-axis at the point Q. [3]

Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$.

d. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. [5]

Find the area of the region R.

e. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. [6]

(i) Show that $g(x) \le x - 1,\ x \in \mathbb{R}^{+}$.

(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^{+}$.

The random variable X has probability density function f where

$$f(x) = \begin{cases} kx(x+1)(2-x), & 0 \leqslant x \leqslant 2 \\ 0, & \text{otherwise}. \end{cases}$$

a. Sketch the graph of the function. You are not required to find the coordinates of the maximum. [1]

b. Find the value of k . [5]

Let $f\left(x\right) = \frac{2-3x^5}{2x^3},\ x \in \mathbb{R},\ x \ne 0$.

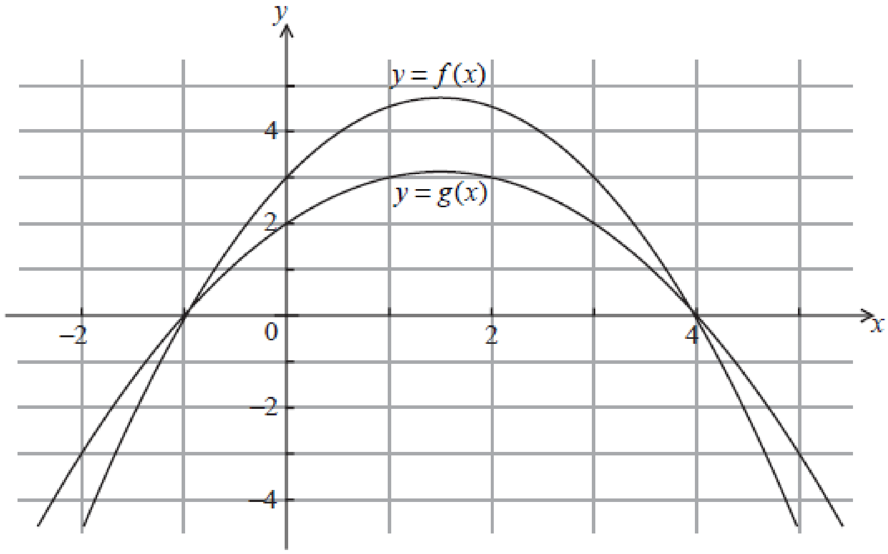
a. The graph of $y = f\left(x\right)$ has a local maximum at A. Find the coordinates of A. [5]

b.i. Show that there is exactly one point of inflexion, B, on the graph of $y = f\left(x\right)$. [5]

b.ii. The coordinates of B can be expressed in the form B$\left(2^{a},\ b \times 2^{-3a}\right)$ where *a*, *b*$\in \mathbb{Q}$. Find the value of *a* and the value of *b*. [3]

c. Sketch the graph of $y = f(x)$ showing clearly the position of the points A and B. [4]

Shown below are the graphs of $y = f(x)$ and $y = g(x)$.



If $(f \circ g)(x) = 3$, find all possible values of x.

The function f is defined on the domain $x \geqslant 0$ by $f(x) = e^x - x^e$.

a. (i) Find an expression for $f'(x)$. [3]

(ii) Given that the equation $f'(x) = 0$ has two roots, state their values.

b. Sketch the graph of f , showing clearly the coordinates of the maximum and minimum. [3]

c. Hence show that $e^\pi > \pi^e$. [1]

The function f is defined on the domain $\left[0, \frac{3\pi}{2}\right]$ by $f(x) = e^{-x} \cos x$.

a. State the two zeros of f . [1]

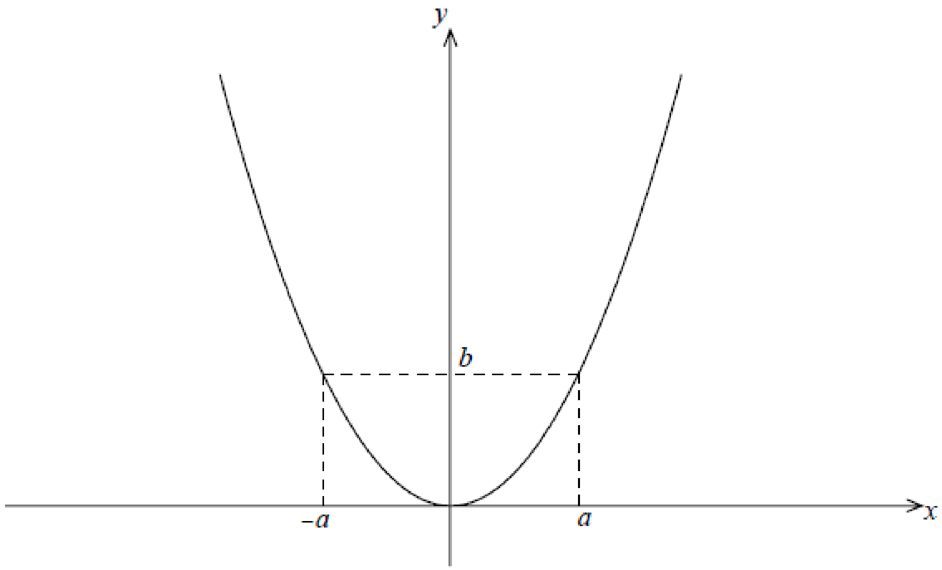
b. Sketch the graph of f . [1]

c. The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B . Show that the ratio of the area of A to the area of B is [7]

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

The diagram below shows the graph of the function $y=f(x)$, defined for all $x\in\mathbb{R}$,

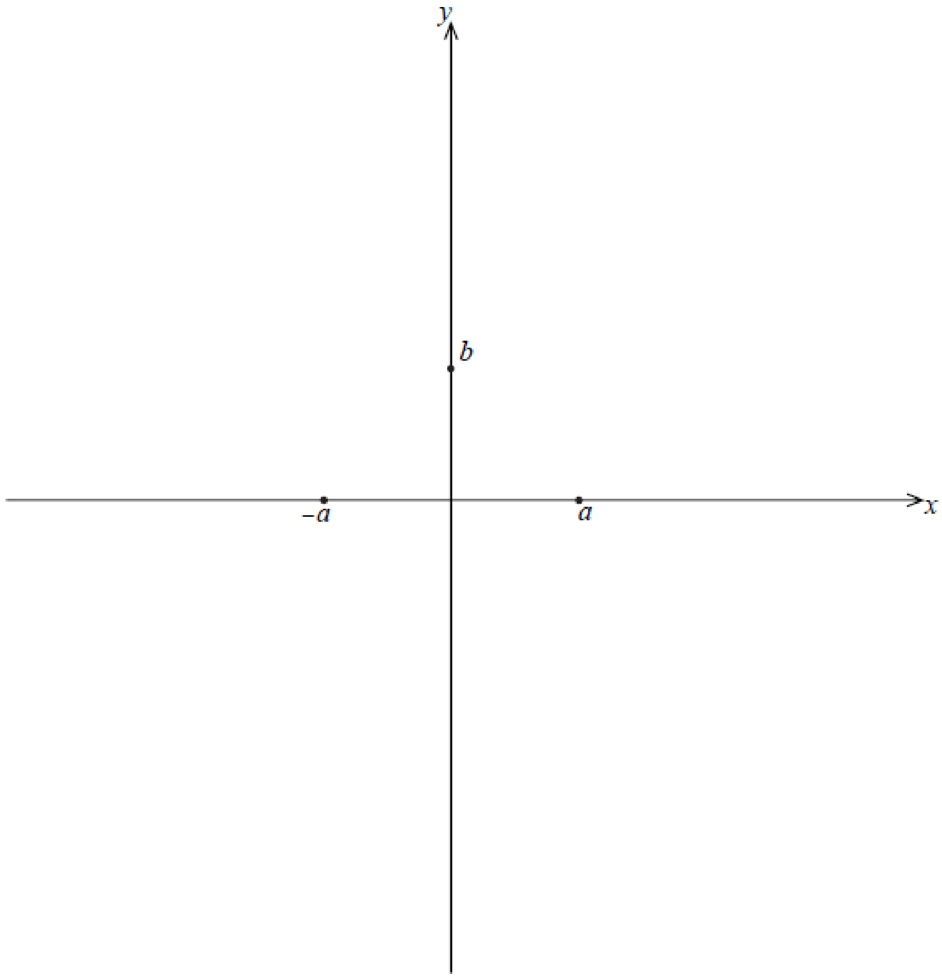
where $b>a>0$.



Consider the function $g(x)=\frac{1}{f(x-a)-b}$.

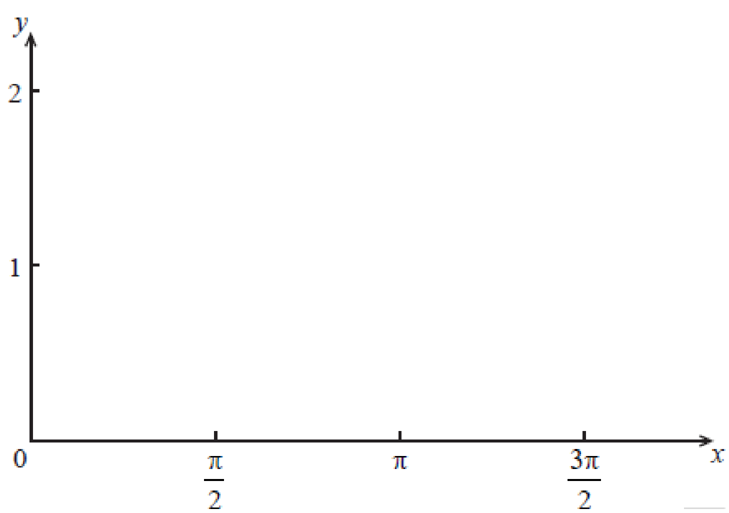
a. Find the largest possible domain of the function g . [2]

b. On the axes below, sketch the graph of $y=g(x)$. On the graph, indicate any asymptotes and local maxima or minima, and write down their equations and coordinates. [6]



Given that $f(x) = 1 + \sin x,\ 0 \leqslant x \leqslant \frac{3\pi}{2}$,

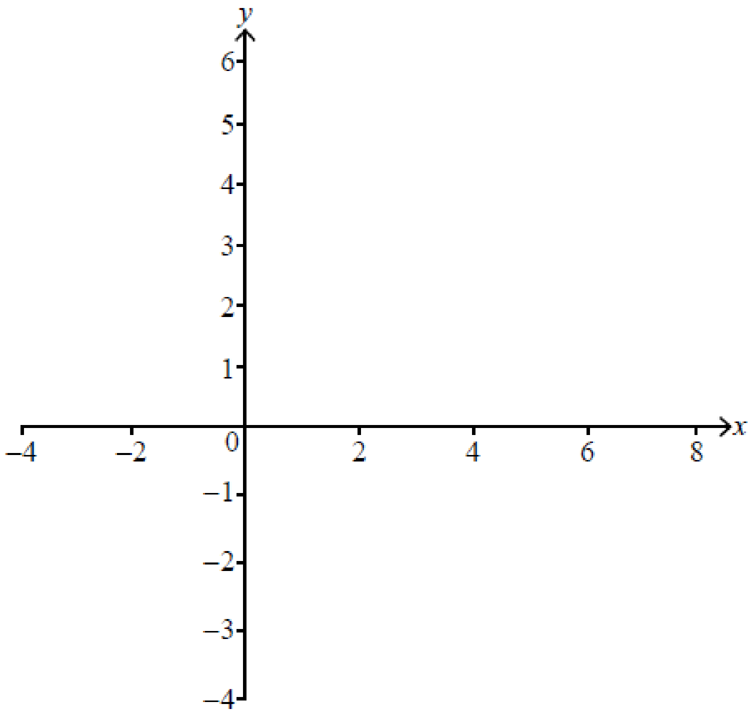
a. sketch the graph of f; [1]



b. show that $(f(x))^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$; [1]

c. find the volume of the solid formed when the graph of f is rotated through 2π radians about the x-axis. [4]

a. Sketch the graphs of $y = \frac{x}{2} + 1$ and $y = |x - 2|$ on the following axes. [3]



b. Solve the equation $\frac{x}{2} + 1 = |x - 2|$. [4]

A function is defined as $f(x) = k\sqrt{x}$, with $k > 0$ and $x \geqslant 0$.

(a) Sketch the graph of $y = f(x)$.

(b) Show that *f* is a one-to-one function.

(c) Find the inverse function, $f^{-1}(x)$ and state its domain.

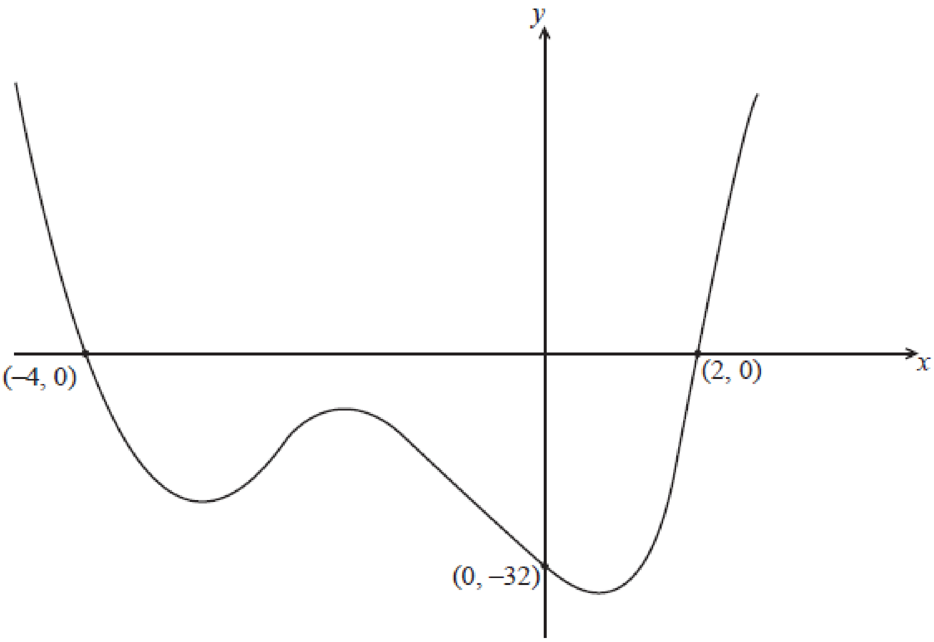
(d) If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at the point (4, 4) find the value of *k* .

(e) Consider the graphs of $y = f(x)$ and $y = f^{-1}(x)$ using the value of *k* found in part (d).

(i) Find the area enclosed by the two graphs.

(ii) The line $x = c$ cuts the graphs of $y = f(x)$ and $y = f^{-1}(x)$ at the points P and Q respectively. Given that the tangent to $y = f(x)$ at point P is parallel to the tangent to $y = f^{-1}(x)$ at point Q find the value of *c* .

The graph of a polynomial function *f* of degree 4 is shown below.



A.a Given that $(x + \mathrm{i}y)^2 = -5 + 12\mathrm{i},\ x,\ y \in \mathbb{R}$. Show that [2]

(i) $x^2 - y^2 = -5$;

(ii) $xy = 6$.

A.b Hence find the two square roots of $-5 + 12\mathrm{i}$. [5]

A.c For any complex number z , show that $(z^*)^2 = (z^2)^*$. [3]

A.d Hence write down the two square roots of $-5 - 12\mathrm{i}$. [2]

B.a Explain why, of the four roots of the equation $f(x) = 0$, two are real and two are complex. [2]

B.b The curve passes through the point $(-1,\ -18)$. Find $f(x)$ in the form [5]

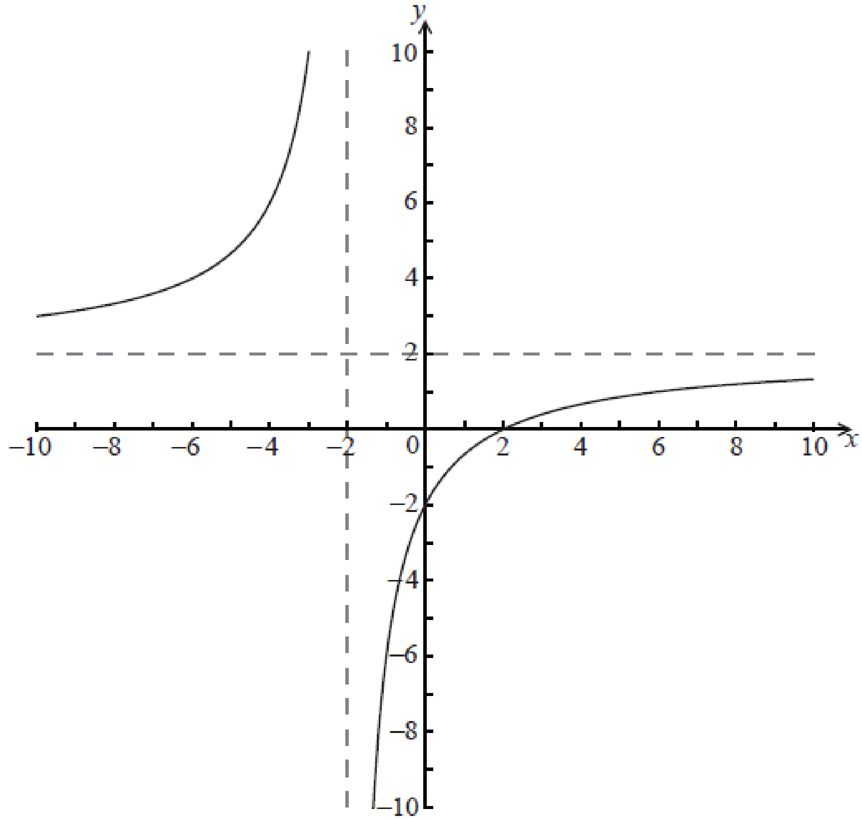
$f(x) = (x - a)(x - b)(x^2 + cx + d),$ where $a,\ b,\ c,\ d \in \mathbb{Z}$.

B.c Find the two complex roots of the equation $f(x) = 0$ in Cartesian form. [2]

B.d Draw the four roots on the complex plane (the Argand diagram). [2]

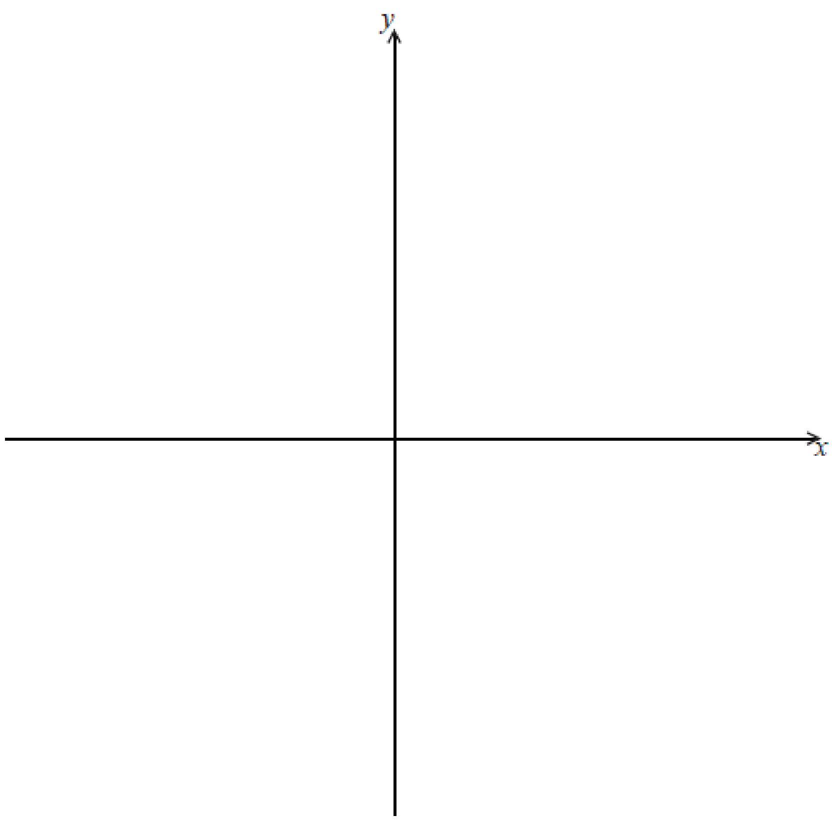
B.e Express each of the four roots of the equation in the form $r\mathrm{e}^{\mathrm{i}\theta}$. [6]

The graph of $y = \frac{a+x}{b+cx}$ is drawn below.



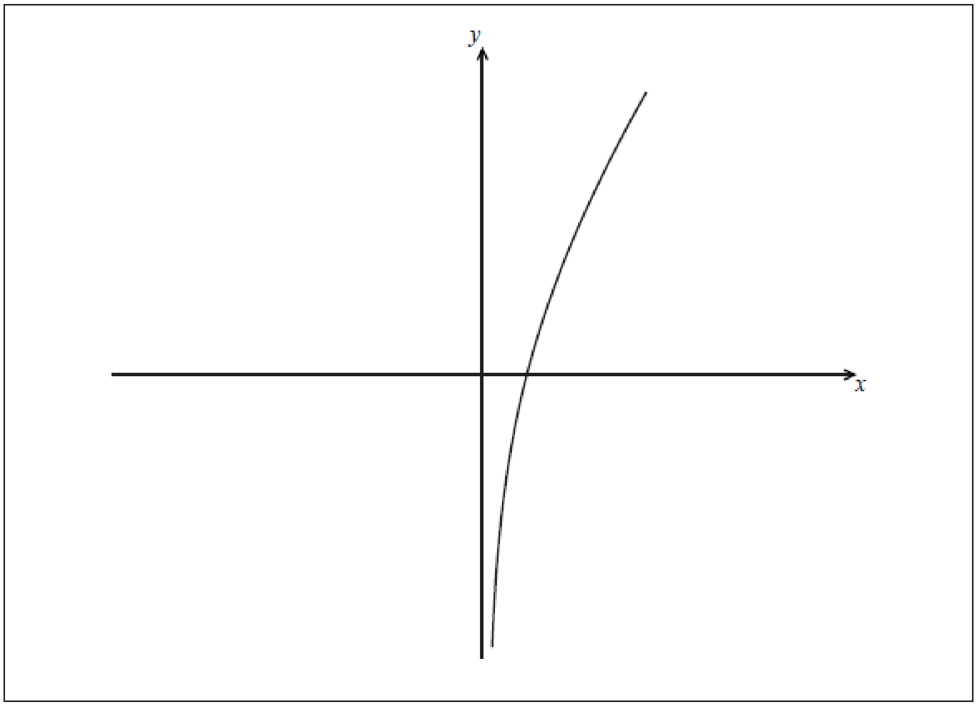
(a) Find the value of a, the value of b and the value of c.

(b) Using the values of a, b and c found in part (a), sketch the graph of $y = \left|\frac{b+cx}{a+x}\right|$ on the axes below, showing clearly all intercepts and asymptotes.



The graph below shows $y = f(x)$, where $f(x) = x + \ln x$.

(a) On the graph below, sketch the curve $y = f^{-1}(x)$.



(b) Find the coordinates of the point of intersection of the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$.

Consider the function $f(x) = \frac{\ln x}{x}$, $0 < x < e^2$.

a. (i) Solve the equation $f'(x) = 0$. [5]

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f .

b. Show that there is a point of inflexion on the graph and determine its coordinates. [5]

c. Sketch the graph of $y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum. [3]

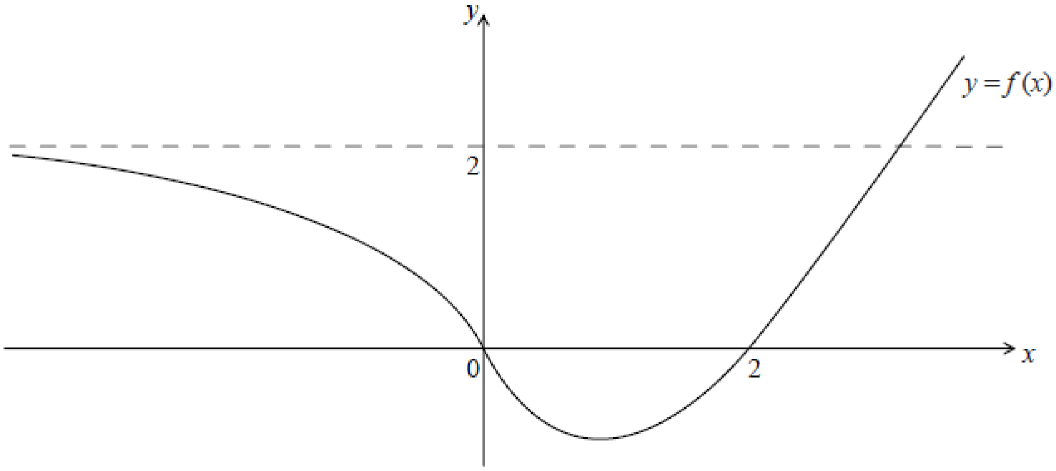
d. Now consider the functions $g(x) = \frac{\ln|x|}{x}$ and $h(x) = \frac{\ln|x|}{|x|}$, where $0 < x < e^2$. [6]

(i) Sketch the graph of $y = g(x)$.

(ii) Write down the range of g .

(iii) Find the values of x such that $h(x) > g(x)$.

The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 2$.



a. Sketch the graph of $y = \frac{1}{f(x)}$. [3]

b. Sketch the graph of $y = x\ f(x)$. [3]

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a. (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \leqslant x \leqslant \frac{\pi}{2}$. [9]

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \leqslant x \leqslant \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

b. Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$. [8]

c. The increasing function *f* satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$. [8]

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

The diagram below shows a solid with volume V , obtained from a cube with edge $a > 1$ when a smaller cube with edge $\frac{1}{a}$ is removed.

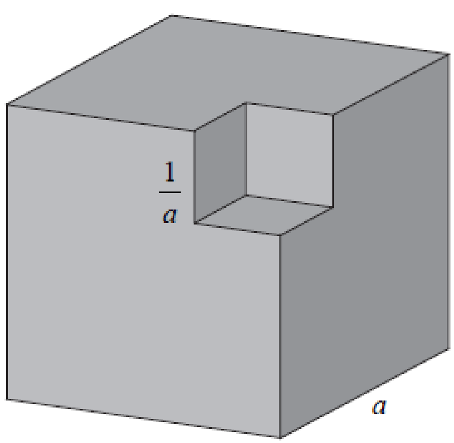


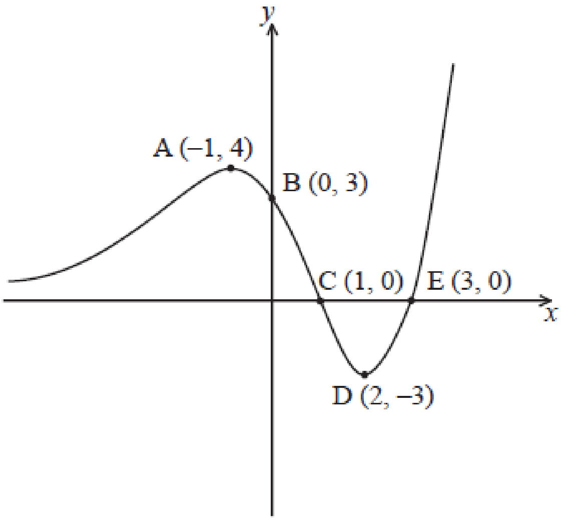
diagram not to scale

Let $x = a - \frac{1}{a}$

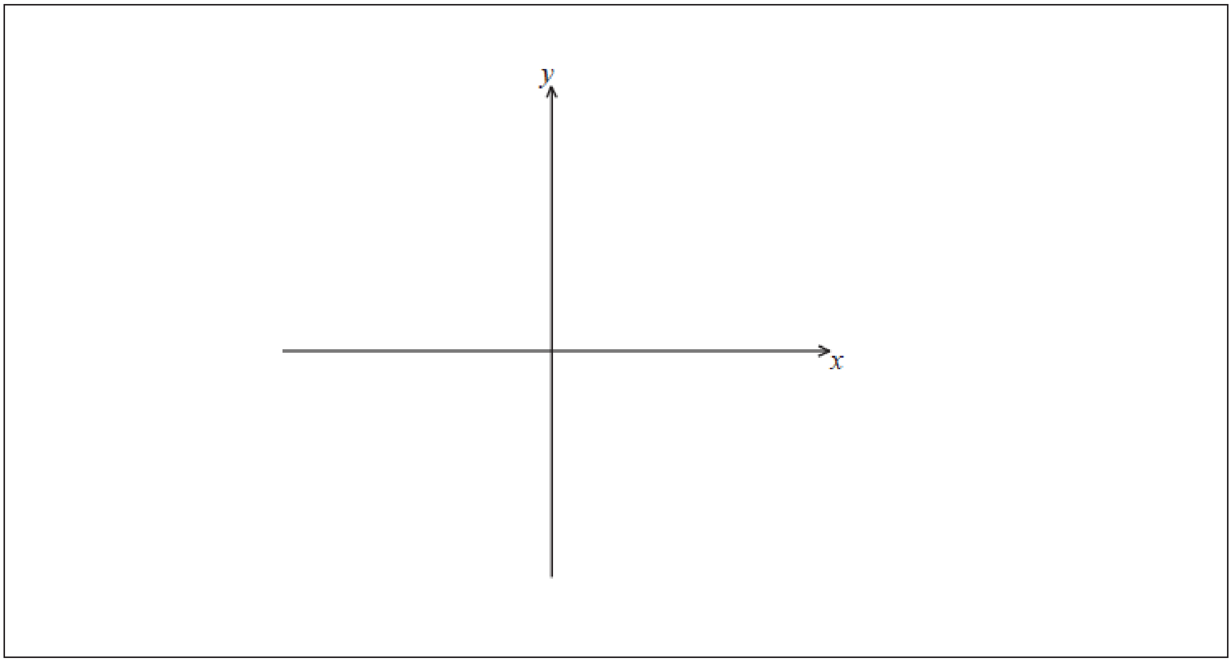
(a) Find V in terms of x .

(b) Hence or otherwise, show that the only value of a for which $V = 4x$ is $a = \frac{1+\sqrt{5}}{2}$.

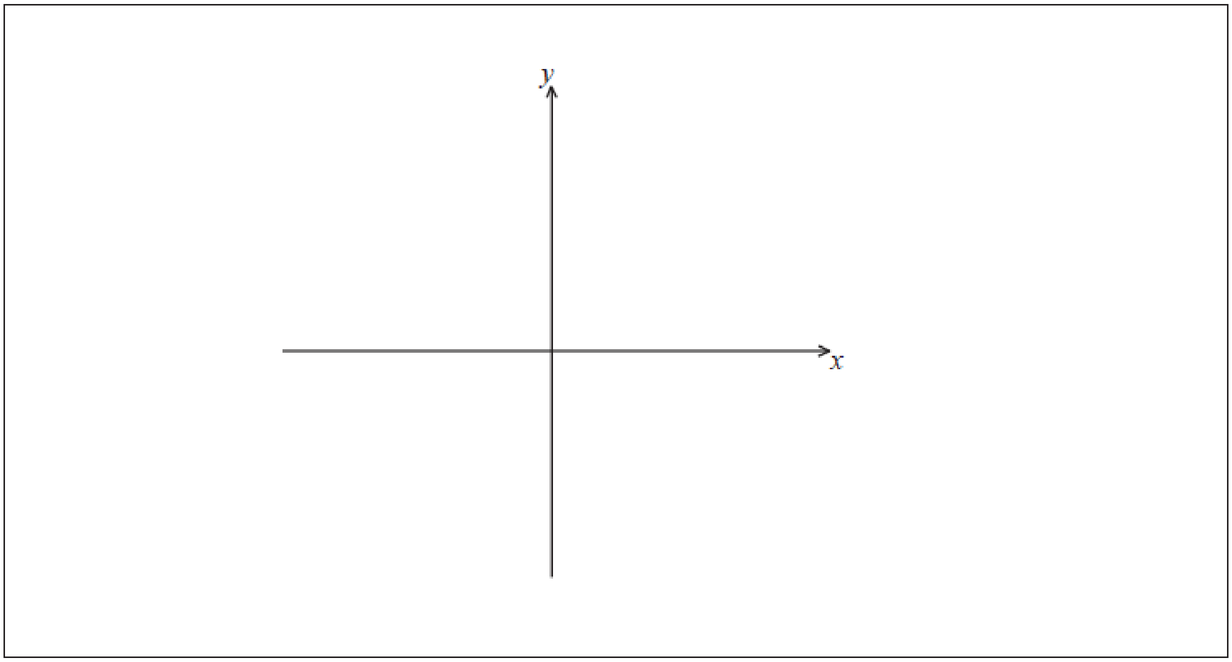
The graph of $y = f(x)$ is shown below, where A is a local maximum point and D is a local minimum point.



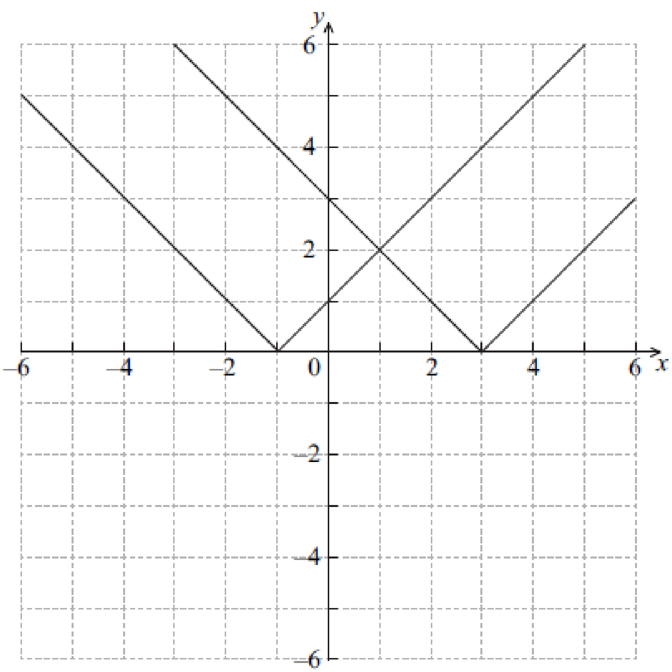
a. On the axes below, sketch the graph of $y = \frac{1}{f(x)}$, clearly showing the coordinates of the images of the points A, B and D, labelling them A', B', and D' respectively, and the equations of any vertical asymptotes. [3]



b. On the axes below, sketch the graph of the derivative $y = f'(x)$, clearly showing the coordinates of the images of the points A and D, labelling them A'' and D'' respectively. [3]

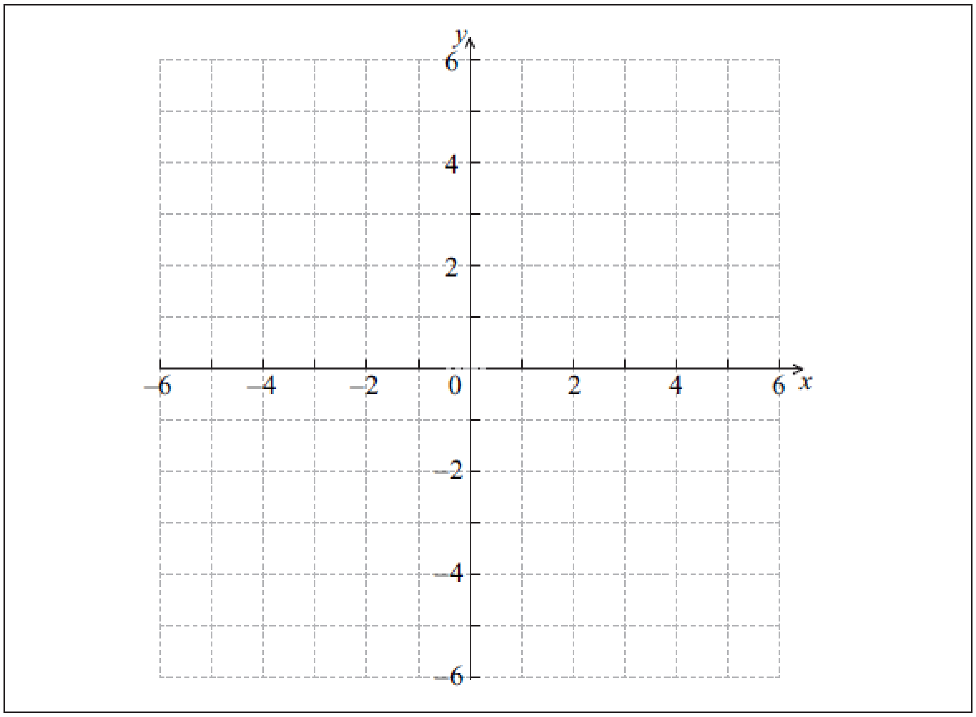


The graphs of $y = |x+1|$ and $y = |x-3|$ are shown below.



Let $f(x) = |x+1| - |x-3|$.

a. Draw the graph of $y = f(x)$ on the blank grid below. [4]



b. Hence state the value of [4]

(i) $f'(-3)$;

(ii) $f'(2.7)$;

(iii) $\int_{-3}^{-2} f(x)dx$.